**Thinking Like a Physicist**

**Casimir Effect**

One of the most startling results in quantum physics is that if you place two neutral mirrors in a vacuum, in the dark at absolute zero, they will be attracted to each other! This attraction is called the Casimir Effect and is one of the most profound and bizarre predictions of 20th century physics. It has ramifications for many areas in physics including nanotechnology and cosmology. Let’s take a closer look at what causes the Casimir Effect.

**Thought Experiment #1―Standing Waves**

Imagine setting up two plane parallel mirrors of area A and separation L.

**A**

**L**

Imagine that light waves, whose plane wavefronts are parallel to the mirrors, are bouncing back and forth between the mirrors forming *standing waves*, much like the standing waves on a violin string fixed at both ends.

1. Sketch the fundamental mode as well as the first and second harmonic of the standing wave between these two mirrors. Express the wavelength of the wave in terms of the separation distance, L.
2. Each one of these modes can be thought of as a system executing simple harmonic motion (like a pendulum or a mass on a spring), whose frequency is related to the wavelength by . This is the universal relation for any wave moving at speed c; in this case c= speed of light. Express the frequency of each of the modes you sketched in #1 in terms of L and the fundamental frequency.
3. Classically, the energy in any one of these vibrational modes is proportional to the square of the amplitude of vibration, and since the amplitude can take any value, the energy can take any values (any real number ≥ 0). According to quantum mechanics, a system executing simple harmonic motion of frequency f cannot have just any energy; its energy can take only *discrete* values given by , where n=0,1,2,… and h is Planck’s constant. Write out the first three allowed energy values.
4. In the case of a standing wave of light between the two mirrors one might be tempted to think this means the *amplitude* of the standing wave is quantized. Actually, the situation is more subtle and interesting. The integer n in refers to the number of photons (“particles of light”) with frequency existing between the two mirrors. For instance, let’s look at the first harmonic mode, whose frequency is . The energy is this mode can be ,, , etc. An energy of corresponds to 0 photons in this mode (n=0). How many photons are associated with and ? What is the spacing between these allowed energy values?
5. Just as a violin string can be vibrating with many harmonics present at the same time (a “superposition” of harmonics), there can be any number of photons of frequencies , , , etc. bouncing between the mirrors at the same time. For example, there may be 3 photons of frequency , 5 photons of , and no photons of higher frequencies. Write out the total light energy between the mirrors in this example, showing that it is an infinite series.
6. Now let’s suppose we remove all of the photons between the mirrors. We can do this by cooling the mirrors to absolute zero and placing them in a perfect vacuum (which includes of course, being “in the dark”). What is the total light energy between the mirrors? Is it zero? Why not?

**So What?**

We’ve seen that the energy between two mirrors in a vacuum at absolute zero in not zero! Each mode of frequency , (, , , etc. ), contributes a so-called “zero point energy” of , which is due to purely quantum mechanical “vacuum” fluctuations of the electromagnetic field.

What is the nature of these vacuum fluctuations? In elementary quantum mechanics we have the Heisenberg uncertainty principle: , which says that a particle cannot *simultaneously* possess a *definite* position () and a *definite* momentum (). In the same way, the electromagnetic field cannot simultaneously possess a definite electric field (=0) and a definite magnetic field (=0). In particular, the electric and magnetic fields cannot simultaneously both have the definite value *zero* (=0 and =0). Even when there are no photons present, the electromagnetic field is still very much alive, continuously fluctuating in a purely quantum mechanical way.

While the energy associated with vacuum fluctuations of each mode is *finite* (,, ,..etc.), the total vacuum energy between the two mirrors ***appears*** to be *infinite*!

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**Thought Experiment #2―The Power of Mathematics**

But is this really so? Let’s see if we can use mathematics to dig a little deeper!

1. Consider the infinite series . The sum of this infinite series is certainly infinite. On the other hand, consider, . Divide 1 by using long division and show that
2. Evaluate the right hand side of this expression for x=2.
3. Evaluate the left hand side of this expression for x=2.
4. It would seem that !!! How can this make sense???

**What’s happening?**

The function is well-defined for all values of except ( is not defined!). By doing the long-hand division we found another way of expressing this function, namely as the infinite series . The problem is, this infinite series “works” only for values of in the range of (note: this can be proved using more advanced mathematics).

Try the example of and compare with

Roughly speaking, given a meaningless (infinite) expression like , we can writes this as and assume it came from a series representation ( of a function that was simply evaluated at a value of outside its range of validity. In other words, we are just dealing with a “bad representation” of a simple function that is actually well-defined for all values of (except ). In this way, we can see that, .

The advanced mathematics involved in this sort of reasoning is called “analytic continuation”.

In a similar way, it can be shown that

(see section on Riemann Zeta function to learn more).

1. Using this and going back to what we saw in part 1 show that .
2. What happens when we reduce the separation between the mirrors?
3. How does this manifest into a force of attraction between the mirrors?

**Thought Experiment #3―Determining the Casimir Force**

Knowing how depends on we can determine the force, . Start with mirrors separated by distance L, so . Then holding one of the mirrors fixed, let the force acting on the other mirror push this mirror a small distance , so that the separation between the mirrors is and the (reduced) vacuum energy is now .

**ΔL**

**L**

**Fc**

**Fc**

1. Work is defined as *force* x *distance*. What is the work done by the force ?
2. The amount of work must equal the change in the vacuum energy. What is the change in vacuum energy?
3. Recall the binomial series: Using this simplify the relation found in #2.
4. Equate the expressions found in #1 and #3 and divide through by . What do you find?

What you found in #4 is close, but not quite right, because we glossed over a few details. 1) The space between the mirrors is a 3-D volume, not the 1-D line appropriate for our violin analogy. Also, the light waves can bounce between the mirrors at all angles, not just horizontally; 2) Light waves can be polarized either vertical or horizontal polarization.

Taking into account these facts it can be shown (using calculus) that the Casimir force is actually proportional to the surface area of the mirrors and inversely proportional to the fourth power of the mirror separation:

This force has been confirmed experimentally, and represents one of the most remarkable discoveries of 20th century theoretical physics!

**The Riemann Zeta Function**

Rewrite as

This means that

Compare this with the Riemann Zeta function:

So, . Now, just as the infinite series expansion for is valid only for , the infinite series expansion for is valid only for .

Consider the example :

Nevertheless, itself is well-defined for all (just as is well define for all ). In particular, it is well-defined for (we are interested in ).

To figure out what is we use some advanced mathematics that tell us that:

Set to get:

Here is called the Gamma function. When is a positive integer then

For example,

Therefore,

And (result quoted above).

Hence,